# INTEGRABLE HAMILTONIAN SYSTEMS ON LIE GROUPS

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## 1. Abstract

This lecture will be devoted to a class of left invariant variational problems on a Lie group $G$ whose Lie algebra $\mathfrak{g}$ admits Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ with the usual Lie algebraic conditions

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}\ ,\ \ [\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p},\ \ [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}.$$

These variational problems are called affine-quadratic. They are defined by an affine control system

$$\frac{dg}{dt} = g(t)(A + u(t), u(t) \in \mathfrak{k}$$

and the functional $\frac{1}{2}\int_0^T \langle Q(u), u\rangle\, dt$. Here $A$ is a fixed element in $\mathfrak{p}$, $\langle\ ,\ \rangle$ is a negative of the Cartan-Killing quadratic form and $Q$ is a linear mapping on $\mathfrak{k}$. The basic question is to determine the parameters $A$ and $Q$ for which the Hamiltonian system associated with this optimal control problem is integrable.

The Maximum Principle of optimal control then identifies the appropriate Hamiltonians $H$ on $\mathfrak{g}$ which recover and clarify the results of Manakov, Trofimov and Mischenko in the theory of integrable systems on Lie algebras . Particular cases provide natural explanations for the J.Moser's work on integrability based on isospectral methods in which C. Newmann's mechanical problem on the sphere and C. L. Jacobi's geodesic problem on an ellipsoid play the central role.